

Stationary coupling method for renewal process in continuous time

application to strong bounds for the convergence rate of the distribution of the regenerative process

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Abstract

We propose a new modification of the coupling method for renewal process in continuous time. We call this modification “*the stationary coupling method*”, and construct it primarily to obtain the bounds for convergence rate of the distribution of the regenerative processes in the total variation metrics. At the same time this modification of the coupling method demonstrates an improvement of the classical result of polynomial convergence rate of the distribution of the regenerative process – in the case of a heavy tail.

keywords Renewal process, Regenerative process, Rate of convergence, Coupling method

subclass MSC 60B10 MSC 60J25 MSC 60K15

1 Introduction: Coupling method and its modifications

This paper proposes a new modification of the coupling method, which we call the *stationary coupling method*.

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Next, we demonstrate how this method may be used to obtain the bounds for the convergence rate of the distribution of the regenerative process to the stationary distribution in the total variation metrics.

Application of the stationary coupling method demonstrates improvement of the classic results about the polynomial convergence rate of the distribution of the regenerative process.

First, we precise a description of a coupling method.

1.1 Coupling method (see [6]).

The coupling method invented by W. Doeblin in [3] ordinarily is used to obtain the bounds of convergence rate of a Markov process to the stationary regime.

Below we give a detailed description of this method.

Suppose that $(X'_t, t \geq 0)$ and $(X''_t, t \geq 0)$ are two versions of Markov process $(X_t, t \geq 0)$ with different initial states $X'_0 = x'$ and $X''_0 = x''$, and with the same transition function; the state space of the process $(X_t, t \geq 0)$ is \mathcal{X} with σ -algebra $\sigma(\mathcal{X})$.

In what follows we introduce

$$\mathcal{P}_t^{x'}(A) \stackrel{\text{def}}{=} \mathbf{P}\{X'_t \in A\}, \quad \mathcal{P}_t^{x''}(A) \stackrel{\text{def}}{=} \mathbf{P}\{X''_t \in A\}$$

for $A \in \sigma(\mathcal{X})$, and let $\tau(x', x'') \stackrel{\text{def}}{=} \inf \{t > 0 : X'_t = X''_t\}$. Then

$$\left| \mathcal{P}_t^{x'}(A) - \mathcal{P}_t^{x''}(A) \right| \leq \mathbf{P} \{ \tau(x', x'') > t \}$$

by the coupling inequality.

The random variable $\tau(x', x'')$ is called *coupling epoch*.

Now suppose that for some positive increasing unbounded function $\varphi(t)$ we have $\mathbf{E} \varphi(\tau(x', x'')) = C(x', x'') < \infty$. Then from Markov inequality we deduce:

$$\begin{aligned} \left| \mathcal{P}_t^{x'}(A) - \mathcal{P}_t^{x''}(A) \right| &\leq \mathbf{P} \{ \tau(x', x'') > t \} = \\ &= \mathbf{P} \{ \varphi(\tau(x', x'')) > \varphi(t) \} \leq \frac{\mathbf{E} \varphi(\tau(x', x''))}{\varphi(t)}. \end{aligned} \tag{1}$$

Suppose that the process $(X_t, t \geq 0)$ is ergodic, that is, for all initial states $x \in \mathcal{X}$ the distribution \mathcal{P}_t^x converges weakly to the invariant probability measure \mathcal{P} as $t \rightarrow \infty$, i.e. $\mathcal{P}_t^x \Longrightarrow \mathcal{P}$ as $t \rightarrow \infty$.

Integrating of the inequality (1) with respect to the stationary measure \mathcal{P} we obtain

$$\left| \mathcal{P}_t^{x'}(A) - \mathcal{P}(A) \right| \leq \frac{\int \varphi(\tau(x', x'')) d\mathcal{P}(x'')}{\varphi(t)} = \frac{\mathcal{C}(x')}{\varphi(t)}, \quad (2)$$

and

$$\left\| \mathcal{P}_t^{x'} - \mathcal{P} \right\|_{TV} \leq 2 \frac{\mathcal{C}(x')}{\varphi(t)}.$$

1.1.1 Successful coupling (see [4]) and strong successful coupling.

The original coupling method was most commonly used for the Markov chains, i.e. for random processes in discrete time.

It is required to modify application of the coupling method for random processes in continuous time, since this case suggests $\mathbf{P} \{ \tau(x', x'') < +\infty \} < 1$.

To resolve this problem it was proposed to construct (in a special probability space) the paired stochastic process $(\mathcal{Z}_t, t \geq 0) = ((Z'_t, Z''_t), t \geq 0)$ such that:

1. $X'_t \stackrel{\mathcal{D}}{=} Z'_t$ and $X''_t \stackrel{\mathcal{D}}{=} Z''_t$ for all $t \geq 0$;
2. $\mathbf{P} \{ \tau(Z'_0, Z''_0) < \infty \} = 1$, where $\tau(Z'_0, Z''_0) = \tau(\mathcal{Z}_0) \stackrel{\text{def}}{=} \inf \{ t \geq 0 : Z'_t = Z''_t \}$;
3. $Z'_t = Z''_t$ for all $t \geq \tau(Z'_0, Z''_0)$.

The paired stochastic process $(\mathcal{Z}_t, t \geq 0) = ((Z'_t, Z''_t), t \geq 0)$ which satisfies conditions 1–3 is called *successful coupling* – see [4].

Let us replace condition 2 by the condition

$$2'. \mathbf{E} \tau(Z'_0, Z''_0) < \infty, \text{ where } \tau(Z'_0, Z''_0) = \tau(\mathcal{Z}_0) \stackrel{\text{def}}{=} \inf \{ t \geq 0 : Z'_t = Z''_t \}.$$

We call the paired stochastic process $\mathcal{Z}_t = ((Z'_t, Z''_t), t \geq 0)$ which satisfies conditions 1, 2' and 3 *the strong successful coupling*.

Note that the processes $(Z'_t, t \geq 0)$ and $(Z''_t, t \geq 0)$ can be non-Markov, and its finite-dimensional distributions may differ from the finite-dimensional distributions of $(X'_t, t \geq 0)$ and $(X''_t, t \geq 0)$ respectively; furthermore, generally speaking, the processes $(Z'_t, t \geq 0)$ and $(Z''_t, t \geq 0)$ turn out to be dependent.

Then for all $A \in \sigma(\mathcal{X})$ we use the coupling inequality in the following form:

$$\begin{aligned} |\mathcal{P}_t^{x'}(A) - \mathcal{P}_t^{x''}(A)| &= |\mathbf{P}\{X'_t \in A\} - \mathbf{P}\{X''_t \in A\}| = \\ &= |\mathbf{P}\{Z'_t \in A\} - \mathbf{P}\{Z''_t \in A\}| \leq \mathbf{P}\{\tau(Z'_0, Z''_0) \geq t\} \leq \\ &\leq \frac{\mathbf{E} \varphi(\tau(Z'_0, Z''_0))}{\varphi(t)} \leq \frac{C(Z'_0, Z''_0)}{\varphi(t)}, \end{aligned} \tag{3}$$

where $C(Z'_0, Z''_0) \geq \mathbf{E} \varphi(\tau(Z'_0, Z''_0))$.

As $Z'_0 = X'_0 = x'$ and $Z''_0 = X''_0 = x''$, the right-hand side of the inequality depends only on x' and x'' ; $C(Z'_0, Z''_0) = C(x', x'')$. Hence we can integrate the inequality (3) with respect to the stationary measure \mathcal{P} as in (2):

$$\left| \mathcal{P}_t^{x'}(A) - \mathcal{P}(A) \right| \leq \frac{\int_{\mathcal{X}} C(x', x'') \mathcal{P}(dx'')}{\varphi(t)} = \frac{\mathcal{C}(x')}{\varphi(t)},$$

and therefore

$$\left\| \mathcal{P}_t^{x'} - \mathcal{P} \right\|_{TV} \leq 2(\varphi(t))^{-1} \mathcal{C}(x').$$

However, this integration leads to certain difficulties – see, e.g., [8, 9, 10, 11].

1.2 Stationary coupling method.

In what follows we construct a strong successful coupling $(\mathcal{Z}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$ for the process $(X_t, t \geq 0)$ with an initial state $x \in \mathcal{X}$ and its stationary version $(\tilde{X}_t, t \geq 0)$. After that we obtain the estimate for the random variable

$$\tilde{\tau}(x) = \tilde{\tau}(Z_0) \stackrel{\text{def}}{=} \inf \left\{ t > 0 : Z_t = \tilde{Z}_t \right\}.$$

If we prove the finiteness of $\mathbf{E} \varphi(\tilde{\tau}(x))$ then we obtain

$$\|\mathcal{P}_t^x(A) - \mathcal{P}(A)\|_{TV} \leq 2 \mathbf{P}\{\tilde{\tau}(x) > t\} \leq 2 (\varphi(t))^{-1} \mathbf{E} \varphi(\tilde{\tau}(x))$$

analogously to the inequality (3).

Definition 1 (Stationary Coupling).

A successful coupling of the Markov process and its stationary version is called *stationary successful coupling*.

We call the method of construction of stationary successful coupling, and the use of this construction for bounds of the convergence rate of the distribution of a Markov process to the stationary distribution the *stationary coupling method*.

Our goal is to describe the construction of the stationary successful coupling for renewal process, and application of this construction in order to obtain the bounds for the convergence of the distribution of the regenerative processes to the stationary distribution.

1.3 The structure of the article

This article is divided into 5 Sections, including the Introduction.

In Section 2 we set up main definitions and some necessary denotations.

Section 3 describes the construction of the stationary successful coupling for the backward renewal process when *Key Condition* is satisfied.

Section 4 demonstrates application of the stationary coupling method for the bounds of the convergence rate of the backward renewal process.

Section 5 extends the results of Section 4 to the regenerative Markov and regenerative non-Markov processes and discusses the way to use the stationary coupling method for the queueing theory.

2 Some definitions and denotations

2.1 Definitions

Definition 2 (Renewal Process). Let $\{\zeta_i\}_{i=0}^{\infty}$ be a sequence of positive independent random variables, and the random variables $\{\zeta_i\}_{i=1}^{\infty}$ are identically distributed; denote by $F(s) \stackrel{\text{def}}{=} \mathbf{P}\{\zeta_i \leq s\}$ for $i \geq 1$, and by $G(s) \stackrel{\text{def}}{=} \mathbf{P}\{\zeta_0 \leq s\}$.

Suppose that $\mathbf{E} \zeta_i < \infty$ for all $i \geq 0$, and $\theta_n \stackrel{\text{def}}{=} \sum_{i=0}^n \zeta_i$ for each $n \geq 0$.

Each θ_n is referred to as the n^{th} renewal time (or renewal point), the intervals $[\theta_n, \theta_{n+1}]$ being called renewal intervals, and $\{\zeta_i \stackrel{\text{def}}{=} \theta_{i+1} - \theta_i\}_{i=0}^{\infty}$ being called renewal periods; $\theta_0 = \zeta_0$ is called first renewal point.

Then the random variable $R_t \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathbf{1}(\theta_n \leq t) = \max \{n : \theta_n \leq t\}$

(where $\mathbf{1}(\cdot)$ is the indicator function) represents the number of jumps that have occurred by time t , and we call the process $(R_t, t \geq 0)$ a renewal process.

If $\theta_0 = \zeta_0 \neq 0$ then the process $(R_t, t \geq 0)$ is called delayed.

Remark 1. The renewal process $(R_t, t \geq 0)$ is a counting process, and it is not regenerative.

Definition 3 (Backward and Forward Renewal Processes).

Let $N_t \stackrel{\text{def}}{=} (t - \max\{\theta_n : \theta_n \leq t\})$ and $N_t^* \stackrel{\text{def}}{=} (\min\{\theta_n : \theta_n \leq t\} - t)$, where N_t is the backward renewal time of the renewal process R_t , and N_t^* is the forward renewal time of the renewal process R_t .

We call the processes $(N_t, t \geq 0)$ and $(N_t^*, t \geq 0)$ *backward renewal process* and *forward renewal process* respectively.

At the same time we call the process $(N_t, t \geq 0)$ an *embedded backward renewal process* of the renewal process $(R_t, t \geq 0)$.

Remark 2. The processes $(N_t, t \geq 0)$ and $(N_t^*, t \geq 0)$ are Markov piecewise-linear regenerative processes with the state space $\mathcal{R} \stackrel{\text{def}}{=} \mathbf{R}_{\geq 0}$ with the Borel σ -algebra $\sigma(\mathcal{R})$.

Therefore we construct the stationary successful coupling for the backward renewal process $(N_t, t \geq 0)$.

Remark 3. It is a well-known fact that if the distribution $F(s)$ is not lattice then

$$\lim_{t \rightarrow \infty} \mathbf{P}\{N_t \leq s\} = \lim_{t \rightarrow \infty} \mathbf{P}\{N_t^* \leq s\} = \tilde{F}(s),$$

where

$$\tilde{F}(s) = (\mathbf{E} \zeta_1)^{-1} \int_0^s (1 - F(u)) \, \mathrm{d} u. \quad (4)$$

Below we shall see that application of the stationary coupling method is possible only if the following *Key Condition* is satisfied.

Key Condition. In what follows we suppose that the following inequality for the cumulative distribution function of the renewal period of the renewal process (or of the length of the regeneration period of the regenerative process) is true, i.e.

$$\int_{\{s: \exists F'(s)\}} F'(s) \, ds > 0,$$

and $\mathbf{E} \zeta_i < \infty$.

Remark 4. *Key Condition* for the renewal process implies:

- $\mathbf{E} \zeta_i > 0$ for $i \geq 0$;
- There exists an invariant probability distribution \mathcal{P} on $(\mathcal{R}, \sigma(\mathcal{R}))$ which satisfies (4) such that $\mathcal{P}_t^r \implies \mathcal{P}$, where $\mathcal{P}_t^r(M) \stackrel{\text{def}}{=} \mathbf{P}\{N_t \in M \mid N_0 = r\}$, $M \in \sigma(\mathcal{R})$.

Remark 5. For convenience of the reader we assume that the first renewal time of the process $(R_t, t \geq 0)$ has the cumulative distribution function $F_r(s) \stackrel{\text{def}}{=} \frac{F(r+s) - F(r)}{1 - F(r)}$, where $r \geq 0$ is the initial state of the backward renewal process $(N_t, t \geq 0)$.

$F_r(s)$ is a cumulative distribution function of the residual time of the renewal period if r is a given elapsed time of this period.

2.2 Denotations

Denotation 1. For nondecreasing function $F(s)$ we introduce

$$F^{-1}(y) \stackrel{\text{def}}{=} \inf\{x : F(x) \geq y\}.$$

Denotation 2. In what follows $\mu \stackrel{\text{def}}{=} \mathbf{E} \zeta_1$ and $\mu_0 \stackrel{\text{def}}{=} \mathbf{E} \zeta_0$.

Denotation 3. Here and hereafter we put $\tilde{F}(s) \stackrel{\text{def}}{=} \mu^{-1} \int_0^s (1 - F(u)) \, du$ and

$$\tilde{f}(s) \stackrel{\text{def}}{=} \tilde{F}'(s) = \mu^{-1}(1 - F(s)).$$

Denotation 4. For $r \geq 0$ we denote by $F_r(s) \stackrel{\text{def}}{=} \frac{F(s+r) - F(r)}{1 - F(r)}$.

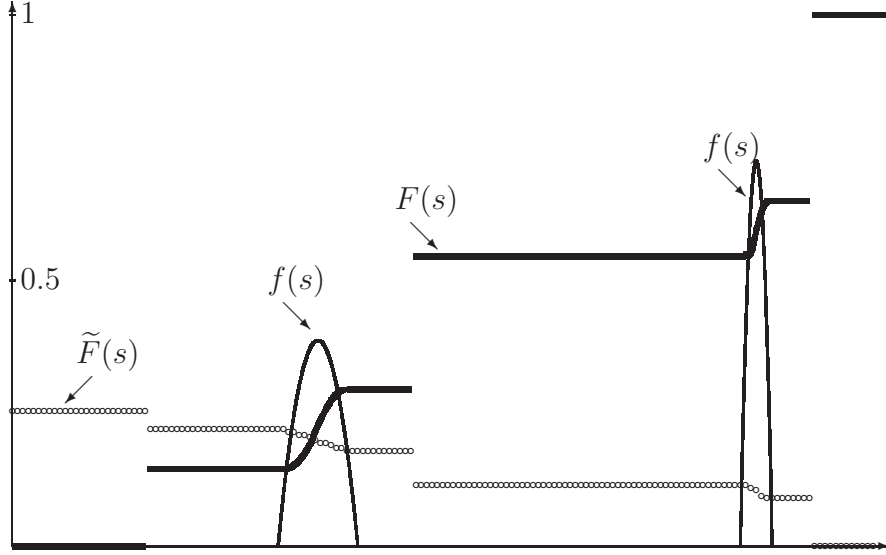


Figure 1: Illustration to Proposition 1 and Remark 6. The mixed type cumulative distribution function $F(s)$ has the positive density $f(s)$ within two intervals, and three jumps.

Denotation 5. $\mathcal{U}, \mathcal{U}', \mathcal{U}'', \mathcal{U}_i, \mathcal{U}'_i, \mathcal{U}''_i, \mathcal{U}'''_i$ are independent uniformly distributed on $[0, 1)$ random variables on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Denotation 6. Denote

$$\begin{aligned} \varphi(s) &\stackrel{\text{def}}{=} \mathbf{1}(\exists F'(s)) \times (F'(s) \wedge \tilde{F}'(s)) = \\ &= \begin{cases} F'(s) \wedge \tilde{F}'(s), & \text{if there exists } F'(s), \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\Phi(s) \stackrel{\text{def}}{=} \int_0^s \varphi(u) \, du; \quad \kappa \stackrel{\text{def}}{=} \Phi(+\infty); \quad \bar{\kappa} \stackrel{\text{def}}{=} 1 - \kappa.$$

Proposition 1. *Key Condition implies $\kappa = \int_0^\infty \varphi(s) \, ds > 0$.*

Proof. Indeed, *Key Condition* implies that there exists a positive density $f(s) = F'(s)$ on some interval (s_1, s_2) , $s_1 < s_2$.

It is easy to check that for all $s \in (0, s_2)$ the inequality $F(s) < 1$ is true. Hence $\tilde{f}(s) = \tilde{F}'(s) = \frac{1 - F(s)}{\mu} \geq \frac{1 - F(s_2)}{\mu} > 0$ for all $s \in (s_1, s_2)$ and

$$\kappa \geq \int_{s_1}^{s_2} \left(f(s) \wedge \tilde{f}(s) \right) ds > 0 - \text{see Fig. 1 for details.}$$

So, Proposition 1 is proved. •

□

Remark 6. If the distribution $F(s)$ which has an absolutely continuous component is close to a discrete distribution then κ is close to zero – see Fig. 1.

Denotation 7. We introduce $\Psi(s) \stackrel{\text{def}}{=} F(s) - \Phi(s)$, $\tilde{\Psi}(s) \stackrel{\text{def}}{=} \tilde{F}(s) - \Phi(s)$.

Remark 7. Note that $\Psi(+\infty) = \tilde{\Psi}(+\infty) = 1 - \kappa [= \bar{\kappa}]$, and the functions $\Phi(s)$, $\Psi(s)$ and $\tilde{\Psi}(s)$ are nondecreasing.

Remark 8. It is easily seen that $\kappa^{-1}\Phi(s)$ is the cumulative distribution function. Also if $\kappa < 1$ then $\bar{\kappa}^{-1}\Psi(s)$ and $\bar{\kappa}^{-1}\tilde{\Psi}(s)$ are the cumulative distribution functions.

If $\kappa = 1$ then $\Phi(s) \equiv F(s) \equiv \tilde{F}(s) = 1 - e^{-\lambda s}$ for $\lambda = \mu^{-1}$ and $\Psi(s) \equiv \tilde{\Psi}(s) \equiv 0$. In this case we put $\bar{\kappa}^{-1}\Psi(s) \stackrel{\text{def}}{=} \bar{\kappa}^{-1}\tilde{\Psi}(s) \stackrel{\text{def}}{=} 0$ and $\Psi^{-1}(u) \stackrel{\text{def}}{=} \tilde{\Psi}^{-1}(u) \stackrel{\text{def}}{=} 0$.

Denotation 8. Here and hereafter let us introduce

$$\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \stackrel{\text{def}}{=} \mathbf{1}(\mathcal{U} < \kappa) \Phi^{-1}(\kappa \mathcal{U}') + \mathbf{1}(\mathcal{U} \geq \kappa) \Psi^{-1}(\bar{\kappa} \mathcal{U}'');$$

$$\tilde{\Xi}(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \stackrel{\text{def}}{=} \mathbf{1}(\mathcal{U} < \kappa) \Phi^{-1}(\kappa \mathcal{U}') + \mathbf{1}(\mathcal{U} \geq \kappa) \tilde{\Psi}^{-1}(\bar{\kappa} \mathcal{U}'').$$

Remark 9. Clearly,

$$F(s) = \kappa \left(\kappa^{-1} \Phi(s) \right) + \bar{\kappa} \left(\bar{\kappa}^{-1} \Psi(s) \right) = \Phi(s) + \Psi(s),$$

and

$$\tilde{F}(s) = \kappa \left(\kappa^{-1} \Phi(s) \right) + \bar{\kappa} \left(\bar{\kappa}^{-1} \tilde{\Psi}(s) \right) = \Phi(s) + \tilde{\Psi}(s).$$

Hence,

$$\begin{aligned}
& \mathbf{P}\{\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \leq s\} = \mathbf{P}\{\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \leq s | \mathcal{U} < \kappa\} \mathbf{P}\{\mathcal{U} < \kappa\} + \\
& + \mathbf{P}\{\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \leq s | \mathcal{U} \geq \kappa\} \mathbf{P}\{\mathcal{U} \geq \kappa\} = \\
& = \kappa \mathbf{P}\{\Phi^{-1}(\kappa \mathcal{U}') \leq s\} + (1 - \kappa) \mathbf{P}\{\Phi^{-1}((1 - \kappa) \mathcal{U}'') \leq s\} = \\
& = \kappa \mathbf{P}\{\mathcal{U}' \leq \Phi(s) \kappa^{-1}\} + (1 - \kappa) \mathbf{P}\{\mathcal{U}'' \leq \Phi(s) (1 - \kappa)^{-1}\} = F(s).
\end{aligned}$$

Analogously, $\mathbf{P}\{\tilde{\Xi}(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \leq s\} = \tilde{F}(s)$.

Moreover, $\mathbf{P}\{\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') = \tilde{\Xi}(\mathcal{U}, \mathcal{U}', \mathcal{U}'')\} = \mathbf{P}\{\mathcal{U} < \kappa\} = \kappa$, since the distribution $\tilde{\Psi}(s)$ is absolutely continuous, and the measure of common part of distributions $\Psi(s)$ and $\tilde{\Psi}(s)$ is equal to zero.

Denotation 9. For the random process $(X_t, t \geq 0)$ we denote by $\mathcal{P}_t^x(M) \stackrel{\text{def}}{=} \mathbf{P}\{X_t \in M | X_0 = x\}$. If this process is ergodic then $\mathcal{P}(M) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \mathcal{P}_t^x(M)$.

3 Stationary successful coupling for the backward renewal process $(N_t, t \geq 0)$.

This section considers the renewal process $(R_t, t \geq 0)$ and its embedded backward renewal process $(N_t, t \geq 0)$; from then on we assume that *Key Condition* is satisfied.

3.1 Construction of the independent versions of a non-stationary and stationary backward renewal process.

At the beginning, let us recall that the *independent* versions of the processes $(N_t, t \geq 0)$ and $(\tilde{N}_t, t \geq 0)$ can be constructed as follows (see, e.g., [1, Chap.V, Proposition 3.5 and Corollary 3.6]).

3.1.1 Construction of the version of the non-stationary backward renewal process $(N_t, t \geq 0)$ – see Fig. 2.

If $N_0 = r$ then $\mathbf{P}\{\zeta_0 \leq s\} = F_r(s)$, where ζ_0 is the first renewal time of the corresponding renewal process $(R_t, t \geq 0)$.

We introduce variables $\zeta_0 \stackrel{\text{def}}{=} F_r^{-1}(\mathcal{U}_0)$, and $\zeta_i \stackrel{\text{def}}{=} F^{-1}(\mathcal{U}_i)$ for $i > 0$; $\theta_i \stackrel{\text{def}}{=} \sum_{j=0}^i \zeta_j$; then

$$\begin{aligned} Z_t &\stackrel{\text{def}}{=} \mathbf{1}(t \geq \theta_0)(t - \max\{\theta_i : \theta_i \leq t\}) + \mathbf{1}(t < \theta_0)(r + t) = \\ &= \mathbf{1}(t \geq \theta_0)(t - \theta_{R_t}) + \mathbf{1}(t < \theta_0)(r + t) \stackrel{\mathcal{D}}{=} N_t. \end{aligned}$$

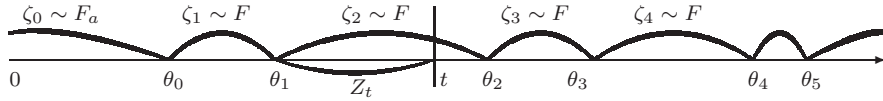


Figure 2: Construction of the version $(Z_t, t \geq 0)$ of the process $(N_t, t \geq 0)$.

3.1.2 Construction of the version of the stationary backward renewal process $(\tilde{N}_t, t \geq 0)$ – see Fig. 3.

Remark 3 provides us with the formula of the distribution of the stationary processes $(\tilde{N}_t, t \geq 0)$ and $(\tilde{N}_t^*, t \geq 0)$: $\mathbf{P}\{\tilde{N}_t \leq s\} = \mathbf{P}\{\tilde{N}_t^* \leq s\} = \tilde{F}(s)$, and therefore $\mathbf{P}\{\tilde{N}_0 \leq s\} = \mathbf{P}\{\tilde{N}_0^* \leq s\} = \tilde{F}(s)$.

So, we put $\tilde{\theta}_0 [= \tilde{N}_0^*] = \tilde{\zeta}_0 \stackrel{\text{def}}{=} \tilde{F}^{-1}(\mathcal{U}'_1)$, and $\tilde{\zeta}_i \stackrel{\text{def}}{=} F^{-1}(\mathcal{U}'_i)$ for $i > 0$; $\tilde{\theta}_i \stackrel{\text{def}}{=} \sum_{j=0}^i \tilde{\zeta}_j$; $\tilde{Z}_0 \stackrel{\text{def}}{=} F_{\tilde{\theta}_0}^{-1}(\mathcal{U}''_1)$ (see Denotation 4); then we put

$$\tilde{Z}_t \stackrel{\text{def}}{=} \mathbf{1}(t < \tilde{\theta}_0) \left(t + \tilde{Z}_0 \right) + \mathbf{1}(t \geq \tilde{\theta}_0) \left(t - \max\{\tilde{\theta}_n : \tilde{\theta}_n \leq t\} \right) \stackrel{\mathcal{D}}{=} \tilde{N}_t.$$

Remark 10. $\mathbf{P}\{\tilde{Z}_0 \leq s\} =$

$$\begin{aligned}
&= \int_0^\infty F_u(s) d\tilde{F}(u) = \int_0^\infty \frac{F(s+u) - F(u)}{1 - F(u)} \times \frac{1 - F(u)}{\mu} du = \\
&= \mu^{-1} \left(\int_0^\infty (1 - F(u)) du - \int_0^\infty ((1 - F(s+u)) du \right) = \\
&= \mu^{-1} \int_0^s (1 - F(u)) du = \tilde{F}(s).
\end{aligned}$$

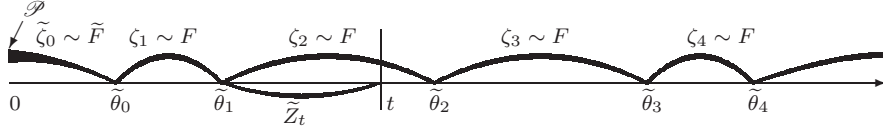


Figure 3: Construction of the version $(\tilde{Z}_t, t \geq 0)$ of the process $(\tilde{N}_t, t \geq 0)$.

Remark 11. The processes $(Z_t, t \geq 0)$ and $(\tilde{Z}_t, t \geq 0)$ described in Sections 3.1.1 and 3.1.2 are *independent* since they are constructed by using independent random variables (see Denotation 5).

3.2 Construction of the stationary successful coupling for the backward renewal process.

Now we construct the successful coupling for the process $(N_t, t \geq 0)$ with the initial state $N_0 = r$ and and its stationary version $(\tilde{N}_t, t \geq 0)$ with the initial distribution $\mathbf{P}\{\tilde{N}_t \leq s\} = \tilde{F}(s)$ (on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ – see Denotation 5).

Here we use the principles of construction exposed in Sections 3.1.1 and 3.1.2. However, the construction considered in these Sections is the construction of the *independent versions* of the processes $(N_t, t \geq 0)$ and $(\tilde{N}_t, t \geq 0)$. We have mentioned before that the successful coupling for the

processes in continuous time is a pair of *dependent* processes. So, we have to modify the construction of Sections 3.1.1 and 3.1.2.

To construct the successful coupling for the processes $(N_t, t \geq 0)$ and $(\tilde{N}_t, t \geq 0)$, i.e. a pair of the *dependent* backward renewal processes, it suffices to construct all renewal times of the corresponding renewal processes $(R_t, t \geq 0)$ and $(\tilde{R}_t, t \geq 0)$ – the times ϑ_i on Fig. 4.

Remark 12. Since the processes $(N_t, t \geq 0)$ and $(\tilde{N}_t, t \geq 0)$ are piecewise-linear processes, the coincidence of these processes can occur only at the common renewal time.

We construct a pair $(\mathcal{Z}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$ by induction – see Fig. 4. Since we assume that studied backward renewal process is a homogeneous Markov process, the distribution of the first renewal time of non-stationary version of this process has the distribution which depends only on the initial state. Namely, $G(s) = F_r(s)$ if $N_0 = r$.

3.2.1 Construction of the process $(\mathcal{Z}_t, t \geq 0)$.

Basis of induction. We put $\theta_0 \stackrel{\text{def}}{=} G^{-1}(\mathcal{U}_0) [= F_r^{-1}(\mathcal{U}_0)]$, $\tilde{\theta}_0 \stackrel{\text{def}}{=} \tilde{F}^{-1}(\mathcal{U}'_0)$, $\tilde{Z}_0 \stackrel{\text{def}}{=} F_{\tilde{\theta}_0}^{-1}(\mathcal{U}''_0)$; here and hereafter θ_0 is the first renewal time of the process $(Z_t, t \geq 0)$, and $\tilde{\theta}_0$ is the first renewal time of the process $(\tilde{Z}_t, t \geq 0)$, \tilde{Z}_0 has an initial distribution \mathcal{P} of the stationary backward renewal process, i.e. $\mathbf{P}\{\tilde{Z}_0 \leq s\} = \tilde{F}(s)$.

Now we introduce $Z_t \stackrel{\text{def}}{=} t + r [= t + N_0]$ and $\tilde{Z}_t \stackrel{\text{def}}{=} t + \tilde{Z}_0$ for $t \in [0, \vartheta_0)$, where $\vartheta_0 \stackrel{\text{def}}{=} t_0 \wedge \tilde{t}_0$ (on Fig. 4: $\vartheta_0 = \tilde{\theta}_0$). Time ϑ_0 is the first time when a renewal of at least one of the processes $(Z_t, t \geq 0)$ and $(\tilde{Z}_t, t \geq 0)$ occurred.

Step of induction. Suppose that we have already constructed the process $(\mathcal{Z}_t, t \geq 0)$ for $t \in [0, \vartheta_n)$, $\vartheta_n = \theta_i \wedge \tilde{\theta}_j$. Then there are only three alternatives.

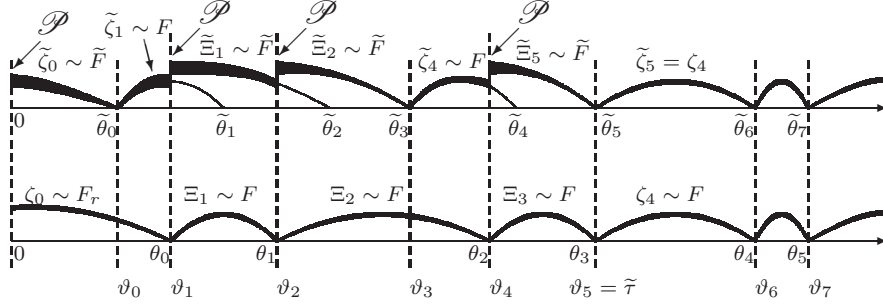


Figure 4: Construction of the successful coupling $(\mathcal{Z}_t, t \geq 0)$.

Case 1. In this case we have $\vartheta_n = \theta_i = \tilde{\theta}_j$ – on Fig. 4 this situation occurs for the first time at the point ϑ_5 , and then at the points ϑ_6 , ϑ_7 , etc.

In this situation at the time ϑ_n the processes coincide, and at the same time they begin a new renewal period with the same distribution.

Then we put

$$Z_{\vartheta_n} = \tilde{Z}_{\vartheta_n} = 0, \quad \theta_{i+1} = \tilde{\theta}_{j+1} = \vartheta_{n+1} = F^{-1}(\mathcal{U}_{n+1}) + \vartheta_n;$$

and $Z_t = \tilde{Z}_t \stackrel{\text{def}}{=} t - \vartheta_n$ for $t \in [\vartheta_n, \vartheta_{n+1})$. Thus after the first coincidence (time $\tilde{\tau} = \vartheta_5$ on Fig. 4) the processes $(Z_t, t \geq 0)$ and $(\tilde{Z}_t, t \geq 0)$ have identical renewal periods, and therefore these processes are identical.

Case 2. In this case we obtain $\vartheta_n = \tilde{\theta}_j < \theta_i$ (the times $\tilde{\theta}_0$ and $\tilde{\theta}_3$ on Fig. 4), i.e. the renewal period of the stationary version of our renewal process ended before the renewal period of the nonstationary version of our renewal process ended. In this case we construct the processes similarly as in the previous Sections, i.e. we put

$$\tilde{Z}_{\vartheta_n} = 0, \quad Z_{\vartheta_n} = Z_{\vartheta_n-0}, \quad \tilde{\theta}_{j+1} \stackrel{\text{def}}{=} \tilde{\theta}_j + F^{-1}(\mathcal{U}_{n+1});$$

and

$$\tilde{Z}_t \stackrel{\text{def}}{=} t - \vartheta_n, \quad Z_t \stackrel{\text{def}}{=} t - \vartheta_n + Z_{\vartheta_n}$$

for $t \in [\vartheta_n, \vartheta_{n+1})$, where $\vartheta_{n+1} \stackrel{\text{def}}{=} \theta_i \wedge \tilde{\theta}_{j+1}$.

In fact, in this situation, we construct the process $(\tilde{Z}_t, t \geq 0)$ according to the scheme of Section 3.1.2, and we do not change anything in the behavior of the process $(Z_t, t \geq 0)$.

Case 3. In this case we result in $\vartheta_n = \theta_i < \tilde{\theta}_j$ (the times θ_0 , θ_1 and θ_2 on Fig. 4): the renewal period of the nonstationary process ended, while the renewal period of the stationary process has not passed yet. In this situation we attempt to continue the behaviour of the processes in such a way that they may coincide with the positive probability (equal to κ) at the next renewal time: we put

$$\theta_{i+1} \stackrel{\text{def}}{=} \theta_i + \Xi(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1}); \quad \tilde{\theta}_j \stackrel{\text{def}}{=} \theta_i + \Xi(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1}),$$

and

$$Z_t \stackrel{\text{def}}{=} t - \vartheta_n, \quad \tilde{Z}_t \stackrel{\text{def}}{=} t - \vartheta_n + F_\varrho^{-1}(\mathcal{U}'''_{n+1})$$

for $t \in [\vartheta_n, \vartheta_{n+1})$, where $\varrho = \tilde{\Xi}(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1})$ and $\vartheta_{n+1} \stackrel{\text{def}}{=} \theta_i \wedge \tilde{\theta}_{j+1}$.

As $\mathbf{P} \left\{ \Xi(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1}) = \tilde{\Xi}(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1}) \right\} = \kappa > 0$ (see Remark 9), in this situation $\mathbf{P} \left\{ \theta_i = \tilde{\theta}_{j+1} \right\} = \kappa > 0$.

3.2.2 The process $\left(\mathcal{Z}_t, t \geq 0 \right) = \left(\left(Z_t, \tilde{Z}_t \right), t \geq 0 \right)$ is a successful coupling for the processes $\left(N_t, t \geq 0 \right)$ and $\left(\tilde{N}_t, t \geq 0 \right)$.

Lemma 1. $Z_t \stackrel{\mathcal{D}}{=} N_t$ and $\tilde{Z}_t \stackrel{\mathcal{D}}{=} \tilde{N}_t$ for all $t \geq 0$.

Proof. First, we see that the construction of the non-stationary process $\left(Z_t, t \geq 0 \right)$ is identical to the construction of Section 3.1.1. So, the process $\left(Z_t, t \geq 0 \right)$ is Markov, and $Z_t \stackrel{\mathcal{D}}{=} N_t$ for all $t \geq 0$.

Now consider the process $\left(\tilde{Z}_t, t \geq 0 \right)$ for $t \in [0, \theta_0)$. Its construction is identical to the construction of Section 3.1.2.

At the time θ_0 , this process restarts from the stationary distribution. At the time θ_0 , we have forgotten the previous history of the process $\left(\tilde{Z}_t, t \geq 0 \right)$, and it began its motion further. The construction of the process $\left(\tilde{Z}_t, t \geq 0 \right)$ after time θ_0 is identical to the construction of Section 3.1.2. Therefore the distribution of this process is stationary as long as we do not interfere with the construction of this process. It means that $\tilde{Z}_t \stackrel{\mathcal{D}}{=} \tilde{N}_t$ between the time θ_0 and θ_1 .

Now we consider the next intervals $[\theta_i, \theta_{i+1})$.

In this case the process $(\tilde{Z}_t, t \geq 0)$ restarts from the stationary distribution at times θ_i , and up to the next restart (at the time θ_{i+1}) this process has a stationary distribution.

So, for all $t \geq 0$ we have $\tilde{Z}_t \stackrel{\mathcal{D}}{=} \tilde{N}_t$ as $\mathbf{E}(\theta_{i+1} - \theta_i) > 0$ (see Remark 4), and Lemma 1 is proved. • \square

Lemma 2. $\mathbf{P}\{\tilde{\tau}(r) < \infty\} = 1$, where $\tilde{\tau}(r) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 : Z_t = \tilde{Z}_t \mid N_0 = r \right\}$, $r \in \mathcal{R}$.

Proof. Denote by $\mathcal{S}_n \stackrel{\text{def}}{=} \left\{ Z_{\theta_n} = \tilde{Z}_{\theta_n} \right\}$,

$$\mathbf{S}_n \stackrel{\text{def}}{=} \left(\mathcal{S}_n \cap \left(\bigcap_{1 \leq i \leq n-1} \overline{\mathcal{S}_i} \right) \right) = \left\{ Z_{\theta_n} = \tilde{Z}_{\theta_n} \text{ \& } Z_{\theta_i} \neq \tilde{Z}_{\theta_i}, i < n \right\}.$$

It can be easily seen that $\mathbf{S}_n \cap \mathbf{S}_m = \emptyset$ if $n \neq m$, $\mathbf{P}(\mathbf{S}_n) = \kappa \bar{\kappa}^{n-1}$, and $\mathbf{P}\left(\bigcup_{n=1}^{\infty} \mathbf{S}_n\right) = 1$.

In accordance with our construction of the pair $(\mathcal{Z}_t, t \geq 0)$, we obtain $\mathbf{P}\left\{Z_{\theta_0} \neq \tilde{Z}_{\theta_0}\right\} = 1$ since the distribution $\tilde{F}(s)$ is absolutely continuous, and $\mathbf{P}\{\tilde{\tau} = \theta_n\} = \mathbf{P}(\mathbf{S}_n) = \kappa \bar{\kappa}^{n-1}$, $n \geq 1$.

It is significant to mention that events \mathbf{S}_n not depend on r , $n \geq 1$.

Note that random variables $\zeta_i \stackrel{\text{def}}{=} \theta_i - \theta_{i-1}$ ($i \geq 1$) are independent since they have been constructed by using different independent random variables.

Then from the law of total probability we deduce:

$$\begin{aligned} \mathbf{P}\{\tilde{\tau}(r) < \infty\} &= \sum_{n=1}^{\infty} \left(\mathbf{P}\{\tilde{\tau}(r) < \infty \mid \mathbf{S}_n\} \mathbf{P}\{\mathbf{S}_n\} \right) = \\ &= \sum_{n=1}^{\infty} \mathbf{P}(\mathbf{S}_n) \left(\mathbf{P}\{\zeta_n < \infty \mid \mathbf{S}_n\} \prod_{1 \leq i \leq n-1} \mathbf{P}\{\zeta_i < \infty \mid \mathbf{S}_n\} \right) = \\ &= \sum_{n=1}^{\infty} \mathbf{P}(\mathbf{S}_n) \left(\kappa^{-1} \Phi(\infty) \prod_{1 \leq i \leq n-1} (\bar{\kappa}^{-1} \Psi(\infty)) \right) = \kappa \sum_{n=1}^{\infty} \bar{\kappa}^{n-1} = 1. \end{aligned}$$

If $\kappa = 1$ then $\mathbf{P}(\mathbf{S}_1) = 1$, and $\tilde{\tau}(r) = \theta_1$; $\mathbf{P}\{\theta_1 < \infty\} = 1$ if *Key Condition* is satisfied.

So, Lemma 2 is proved. • \square

Lemma 3. $Z_t = \tilde{Z}_t$ for all $t \geq \tilde{\tau}(r)$.

Proof. The statement of the Lemma 3 follows from the construction of the Case 1 (Section 3.2.1) and Lemma 3 is proved. \square

Lemma 4. $\mathbf{E} \tilde{\tau}(r) < \infty$.

To prove Lemma 4 we need the following elementary

Proposition 2. Let ξ be a non-negative random variable, and let \mathcal{E}_1 and \mathcal{E}_2 be an events such that the random variables ξ and $\mathbf{1}(\mathcal{E}_2)$ are independent. Then

$$\begin{aligned} \mathbf{E}(\xi \times \mathbf{1}(\mathcal{E}_1)) &\leq \mathbf{E} \xi; \\ \mathbf{E}(\xi \times \mathbf{1}(\mathcal{E}_2)) &= \mathbf{E} \xi \mathbf{P}(\mathcal{E}_2). \end{aligned} \tag{5}$$

of Lemma 4. Let us recall, that $\mathbf{P}(\mathbf{S}_n) = \kappa \bar{\kappa}^n$, $\zeta_i \stackrel{\text{def}}{=} \theta_i - \theta_{i-1}$, and for all $i \neq j$ the random variable ζ_i does not depend on the random event \mathcal{S}_j .

Then from (5) we have for $\kappa \in (0, 1)$:

$$\begin{aligned} \mathbf{E} \tilde{\tau}(r) &= \mathbf{E} \zeta_0 + \mathbf{E}(\mathbf{1}(\mathbf{S}_1) \zeta_1) + \mathbf{E}(\mathbf{1}(\mathbf{S}_2)(\zeta_1 + \zeta_2)) + \\ &\quad + \mathbf{E}(\mathbf{1}(\mathbf{S}_3)(\zeta_1 + \zeta_2 + \zeta_3)) + \dots + \mathbf{E} \left(\mathbf{1}(\mathbf{S}_n) \sum_{i=1}^n \zeta_i \right) + \dots = \\ &= \mathbf{E} \zeta_0 + \sum_{i=1}^{\infty} \left(\mathbf{E} \left(\zeta_i \times \left(\mathbf{1}(\mathbf{S}_i) + \sum_{j=i+1}^{\infty} \mathbf{1}(\mathbf{S}_j) \right) \right) \right) \leq \\ &\leq \mathbf{E} \zeta_0 + \sum_{i=1}^{\infty} \left(\mathbf{E} \zeta_i \times \left(\prod_{\ell=1}^{i-1} \mathbf{P}(\overline{\mathcal{S}}_{\ell}) \times \left(1 + \right. \right. \right. \\ &\quad \left. \left. \left. + \sum_{j=i+1}^{\infty} \left(\prod_{\ell=i+1}^{j-1} \mathbf{P}(\overline{\mathcal{S}}_{\ell}) \prod_{\ell=j}^{\infty} \mathbf{P}(\mathcal{S}_{\ell}) \right) \right) \right) \right) = \\ &= \mathbf{E} \zeta_0 + \sum_{i=1}^{\infty} \left(\mathbf{E} \zeta_i \times \bar{\kappa}^{i-1} \left(1 + \kappa \sum_{j=0}^{\infty} \bar{\kappa}^j \right) \right) = \mathbf{E} \zeta_0 + 2\kappa^{-1} \mathbf{E} \zeta_1, \end{aligned} \tag{6}$$

here we put $\prod_{\ell=k}^{k-1}(\cdot) \stackrel{\text{def}}{=} 1$.

If $\kappa = 1$ then $\mathbf{P}(\mathcal{S}_1) = 1$. Hence $\mathbf{E} \tilde{\tau}(r) = \mathbf{E} \theta_1 < \infty$ and Lemma 4 is proved. • \square

Thus, Lemmata 1–4 imply the following

Theorem 1. *The paired process $(\mathcal{Z}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$ constructed in Section 3.2.1 is a strong successful coupling for the backward renewal process $(N_t, t \geq 0)$.*

4 Estimation of the coupling epoch of the stationary successful coupling of the backward renewal process.

Lemma 5. *Let $(N_t, t \geq 0)$ be a backward renewal process which satisfies Key Condition with the initial state $N_0 = r$, and $\mu_{0,K} \stackrel{\text{def}}{=} \mathbf{E}(\zeta_0)^K < \infty$, $\mu_K \stackrel{\text{def}}{=} \mathbf{E}(\zeta_1)^K < \infty$ for some $K \geq 1$.*

Then for the stationary coupling $(\mathcal{Z}_t, t \geq 0)$ constructed by the schema exposed in Section 3.2.1 for all $k \in [1, K]$ we have

$$\mathbf{E}(\tilde{\tau}(r))^k = C(k, r) < \infty,$$

where $\tilde{\tau}(r) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 : Z_t = \tilde{Z}_t \mid N_0 = r \right\}$, and

$$\begin{aligned} C(k, r) &\leq \mu_{0,k} \sum_{n=1}^{\infty} ((n+1)^{k-1} \bar{\kappa}^{n-1}) + \\ &\quad + \mu_k \sum_{n=1}^{\infty} ((\kappa n(n+2)^{k-1} + \bar{\kappa} (n+1)^{k-1}) \bar{\kappa}^{n-1}) \stackrel{\text{def}}{=} \hat{C}(k, \zeta_0). \end{aligned}$$

Proof. Suppose that $\tilde{\tau}(r) = \theta_\nu$.

Then by (5) for $1 \leq i < \nu$ we obtain

$$\begin{aligned}
EE\left((\zeta_i)^k \mathbf{1}(\mathbf{S}_\nu)\right) &= \mathbf{P}(\mathbf{S}_\nu) \int_0^\infty s^k d\left(\bar{\kappa}^{-1}\Psi(s)\right) \leq \mu_k \bar{\kappa}^{\nu-2} \kappa; \\
\mathbf{E}\left((\zeta_\nu)^k \mathbf{1}(\mathbf{S}_\nu)\right) &= \mathbf{P}(\mathbf{S}_\nu) \int_0^\infty s^k d\left(\kappa^{-1}\Phi(s)\right) \leq \mu_k \bar{\kappa}^\nu; \\
\text{and } \mathbf{E}\left((\zeta_0)^k \mathbf{1}(\mathbf{S}_\nu)\right) &= \kappa \bar{\kappa}^\nu \mu_{0,k}.
\end{aligned} \tag{7}$$

Using the inequalities (7) as well as Jensen's inequality for $k \geq 1$ and $a_i \geq 0$ in the form $\left(\sum_{i=1}^n a_i\right)^k \leq n^{k-1} \sum_{i=1}^n a_i^k$ we result in the expression similar to formula (6):

$$\begin{aligned}
\mathbf{E}(\tilde{\tau}(r))^k &= \mathbf{E}\left(\sum_{n=1}^\infty \left(\mathbf{1}(\mathbf{S}_n) \left(\zeta_0 + \sum_{i=1}^n \zeta_i\right)^k\right)\right) \leq \\
&\leq \mathbf{E}\left(\sum_{n=1}^\infty \left((n+1)^{k-1} \left(\zeta_0^k + \sum_{i=1}^n \zeta_i^k\right) \mathbf{1}(\mathbf{S}_n)\right)\right) = \\
&= \sum_{n=1}^\infty ((n+1)^{k-1} (\mathbf{E}((\zeta_0)^k \mathbf{1}(\mathbf{S}_n)) + \\
&\quad + \sum_{1 \leq i \leq n-1} \mathbf{E}((\zeta_i)^k \mathbf{1}(\mathbf{S}_n)) + \mathbf{E}((\zeta_n)^k \mathbf{1}(\mathbf{S}_n))) \leq \\
&\leq \sum_{n=1}^\infty (n+1)^{k-1} (\kappa \bar{\kappa}^{n-1} \mu_{0,k} + (n-1) \mu_k \bar{\kappa} \bar{\kappa}^{n-1} + \mu_k \bar{\kappa}^n) = \\
&= \mu_{0,k} \kappa \sum_{n=1}^\infty ((n+1)^{k-1} \bar{\kappa}^{n-1}) + \\
&\quad + \mu_k \sum_{n=1}^\infty ((\kappa n(n+2)^{k-1} + \bar{\kappa} (n+1)^{k-1}) \bar{\kappa}^{n-1}).
\end{aligned} \tag{8}$$

So, Lemma 5 is proved. • □

From now on we introduce denotations $\mathbf{E} e^{\alpha\zeta_0} \stackrel{\text{def}}{=} \varepsilon_{0,\alpha}; \quad \mathbf{E} e^{\alpha\zeta_1} \stackrel{\text{def}}{=} \varepsilon_\alpha;$
 $\tilde{\varepsilon}_\alpha \stackrel{\text{def}}{=} \int_0^\infty e^{\alpha s} d\Psi(s).$

Remark 13. If *Key Condition* is satisfied then $\tilde{\varepsilon}_0 = (1 - \kappa) < 1$, and there exists $\beta > 0$ such that $\tilde{\varepsilon}_\beta < 1$.

Lemma 6. *Let $(N_t, t \geq 0)$ be a backward renewal process which satisfies Key Condition with the initial state $N_0 = r$, and*

$$\mathbf{E} e^{a\zeta_0} = \varepsilon_{0,a} < \infty, \quad \mathbf{E} e^{a\zeta_1} = \varepsilon_a < \infty$$

for some $a > 0$.

Suppose $\tilde{\varepsilon}_\beta < 1$ for some $\beta > 0$.

Then for the stationary coupling $(\mathcal{Z}_t, t \geq 0)$ constructed by the schema exposed in Section 3.2.1 for all $\gamma \in (0, \beta)$ we have

$$\mathbf{E} e^{\gamma\tilde{\tau}(r)} = \mathcal{C}(\gamma, r) < \infty,$$

where $\tilde{\tau}(r) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 : Z_t = \tilde{Z}_t \mid N_0 = r \right\}$, and

$$\mathcal{C}(\gamma, r) \leq \frac{\varepsilon_{0,\beta}\varepsilon_\beta}{1 - \tilde{\varepsilon}_\beta} \stackrel{\text{def}}{=} \widehat{\mathcal{C}}(\gamma, \zeta_0).$$

Proof. Once more assume that $\tilde{\tau}(r) = \theta_\nu$. Then for $1 \leq i < \nu$ we obtain

$$\mathbf{E} (e^{\beta\zeta_i} \mathbf{1}(\overline{\mathcal{I}}_i)) = \mathbf{P}(\overline{\mathcal{I}}_i) \int_0^\infty e^{\beta s} d(\bar{\kappa}^{-1}\Psi(s)) = \tilde{\varepsilon}_\beta$$

and

$$\mathbf{E} (e^{\beta\zeta_\nu} \mathbf{1}(\mathcal{I}_\nu)) = \mathbf{P}(\mathcal{I}_\nu) \int_0^\infty e^{\beta s} d(\kappa^{-1}\Phi(s)) \leq \varepsilon_\beta.$$

Hence

$$\begin{aligned}
\mathbf{E}(\tilde{\tau}(r))^k &= \mathbf{E} \left(\sum_{n=1}^{\infty} \left(\mathbf{1}(\mathbf{S}_n) \exp \left(\beta \left(\zeta_0 + \sum_{i=1}^n \zeta_i \right) \right) \right) \right) = \\
&= \mathbf{E} \left(\sum_{n=1}^{\infty} \left(e^{\beta \zeta_0} e^{\beta \zeta_n} \mathbf{1}(\mathcal{S}_n) \prod_{1 \leq i \leq n-1} (e^{\beta \zeta_i} \mathbf{1}(\overline{\mathcal{S}}_i)) \right) \right) \leq \quad (9) \\
&\leq \sum_{n=1}^{\infty} \varepsilon_{0,\beta} \varepsilon_{\beta} (\tilde{\varepsilon}_{\beta})^{n-1} = \frac{\varepsilon_{0,\beta} \varepsilon_{\beta}}{1 - \tilde{\varepsilon}_{\beta}}.
\end{aligned}$$

Lemma 6 is proved. • □

Corollary 1. *Let $(N_t, t \geq 0)$ be a backward renewal process which satisfies Key Condition with the initial state $N_0 = r$, and $\mu_{0,K} \stackrel{\text{def}}{=} \mathbf{E}(\zeta_0)^K < \infty$, $\mu_K \stackrel{\text{def}}{=} \mathbf{E}(\zeta_1)^K < \infty$ for some $K \geq 1$.*

Then for all $t \geq 0$ and every $k \in [1, K]$ we have

$$\|\mathcal{P}_t^r - \mathcal{P}\|_{TV} \leq 2\hat{C}(k, \zeta_0) t^{-k},$$

where $\mathcal{P}_t^r(M) \stackrel{\text{def}}{=} \mathbf{P}\{N_t \in M | N_0 = r\}$, $\mathcal{P}(M) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \mathcal{P}_t^r(M)$, $M \in \mathcal{R}$, and ζ_0 has a cumulative distribution function $F_r(s)$.

Corollary 2. *Let $(N_t, t \geq 0)$ be a backward renewal process which satisfies Key Condition with the initial state $N_0 = r$, and for some $a > 0$*

$$\mathbf{E} e^{a\zeta_0} = \varepsilon_{0,a} < \infty, \quad \mathbf{E} e^{a\zeta_1} = \varepsilon_a < \infty.$$

Suppose $\tilde{\varepsilon}_{\beta} < 1$ for some $\beta > 0$.

Then for all $t \geq 0$ and $\gamma \in (0, \beta)$ we derive an estimate

$$\|\mathcal{P}_t^r - \mathcal{P}\|_{TV} \leq 2\hat{\mathcal{C}}(\gamma, \zeta_0) e^{-\gamma t},$$

where $\mathcal{P}_t^r(M) \stackrel{\text{def}}{=} \mathbf{P}\{N_t \in M | N_0 = r\}$, $\mathcal{P}(M) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \mathcal{P}_t^r(M)$, $M \in \mathcal{R}$, and ζ_0 has a cumulative distribution function $F_r(s)$.

Proof. Corollary 1 and Corollary 2 follow from Lemma 5, Lemma 6 and Section 1.2. □

Remark 14. The Corollary 1 improves the classical result: there exists the constant C such that $\|\mathcal{P}_t^r - \mathcal{P}\|_{TV} \leq C t^{-K+1}$ (see [1, 2, 5, 7])

5 Application of the stationary coupling to strong bounds for the convergence rate of the distribution of the regenerative process

5.1 Introduction to regenerative processes

Let us recall the definition of a regenerative process in continuous time.

Definition 4. Assume that the random process $(X_t, t \geq 0)$ with the state space $(\mathcal{X}, \sigma(\mathcal{X}))$ has continuous time parameter $t \in [0, +\infty)$.

Besides, we suppose that there exists a renewal process $(R_t, t \geq 0)$ with the renewal times $\{\theta_j\}_{j=0}^\infty$; as before, $\theta_i = \sum_{j=0}^i \zeta_j$, the random variables $\{\zeta_j\}_{j=0}^\infty$ are independent, and $\{\zeta_j\}_{j=1}^\infty$ are identically distributed.

Furthermore, the pair of processes $((X_t, R_t), t \geq 0)$ has the following property: for each $n \geq 0$, the post- θ_n process $(X_{\theta_n+t}, t \geq 0)$ is independent of $(\theta_0, \dots, \theta_n)$ (or, equivalently, of ζ_0, \dots, ζ_n) and its distribution does not depend upon n .

Then we call the process $(X_t, t \geq 0)$ *regenerative process*.

We call $(R_n, t \geq 0)$ the *embedded renewal process* and refer to the θ_n as *regeneration points* or *regeneration times*.

The behaviour of the process X_t on the k^{th} cycle $\Theta_k \stackrel{\text{def}}{=} (X_{t+\theta_k}, t \in [0, \zeta_{k+1}])$ is a random element with the state space $D_0(\mathcal{X})$ of \mathcal{X} -valued functions which are right-continuous and have left-hand limits and with finite lifelengths – see [1, Chapter 6]; $\{\Theta_k\}_{k=0}^\infty$ are i.i.d. random elements.

We call the intervals (θ_{t-1}, θ_t) the *regeneration periods*, and we call the random variable ζ_k the *length of the k^{th} regeneration period*.

When t_0 equals 0, $(X_t, t \geq 0)$ is called a *nondelayed* (or *pure*) *regenerative process*. Otherwise, the process is called a *delayed regenerative process*.

5.2 Bounds for the convergence rate of the regenerative process.

This Section is devoted to *Markov* regenerative processes. We attempt to derive the strong bounds (in the total variation metrics) for the convergence rate of this process in the case when the distribution of the regeneration period length satisfies *Key Conditions*.

These bounds can be extended to the case of non-Markov regenerative processes by the following reasons.

Remark 15 (On non-Markov regenerative processes). Every arbitrary non-Markov regenerative process $(X_t, t \geq 0)$ with the state space $(\mathcal{X}, \sigma(\mathcal{X}))$ can be extended to the Markov regenerative process $(\bar{X}_t, t \geq 0)$ with the extended state space $(\bar{\mathcal{X}}, \sigma(\bar{\mathcal{X}}))$.

For instance, we can include in the state X_t for $t \in [\theta_{n-1}, \theta_n)$ full history of the process $(X_t, t \geq 0)$ on the time interval $[\theta_{n-1}, t]$ for markovization of non-Markov regenerative process.

The process $\bar{X}_t \stackrel{\text{def}}{=} \{X_s, s \in [\theta_{n-1}, t] \mid t \in [\theta_{n-1}, \theta_n)\}$ is Markov and regenerative with the extended state space $(\bar{\mathcal{X}}, \sigma(\bar{\mathcal{X}}))$.

Denote by $\bar{\mathcal{P}}_t^{\bar{x}}(\bar{M}) \stackrel{\text{def}}{=} \mathbf{P}\{\bar{X}_t \in \bar{M}\}$ for the process \bar{X}_t with the initial state $\bar{X}_0 = \bar{x}$ and $\bar{M} \in \sigma(\bar{\mathcal{X}})$.

If $\mathbf{E}\zeta_i < \infty$ then $\bar{\mathcal{P}}_t^{\bar{x}} \Rightarrow \bar{\mathcal{P}}$, where $\bar{\mathcal{P}}$ is some stationary probability measure on the state space $(\bar{\mathcal{X}}, \sigma(\bar{\mathcal{X}}))$.

If we prove that $\|\bar{\mathcal{P}}_t^{\bar{x}} - \bar{\mathcal{P}}\|_{TV} \leq \psi(t, \bar{x}) [= \phi(t, \zeta_0)]$ for all $t \geq 0$ then this inequality will be true for the original non-Markov regenerative process X_t :

$$\begin{aligned} \|\mathcal{P}_t^x - \mathcal{P}\|_{TV} &\stackrel{\text{def}}{=} 2 \sup_{M \in \sigma(\mathcal{X})} |\mathcal{P}_t^x(M) - \mathcal{P}(M)| \leq \\ &\leq \|\bar{\mathcal{P}}_t - \bar{\mathcal{P}}\|_{TV} \stackrel{\text{def}}{=} 2 \sup_{\bar{M} \in \sigma(\bar{\mathcal{X}})} |\bar{\mathcal{P}}_t(\bar{M}) - \bar{\mathcal{P}}(\bar{M})| \leq \phi(t, \zeta_0). \end{aligned}$$

Besides, the extension for markovization can be more simple for the queueing non-Markov regenerative process (when considering the structure of this process).

So, we deal with the regenerative Markov process $(X_t, t \geq 0)$ with the state space $(\mathcal{X}, \sigma(\mathcal{X}))$; and let $\{\theta_i\}_{i=0}^\infty$ be its regeneration points. As was mentioned before, $\theta_i = \sum_{k=0}^i \zeta_k$ and $\zeta_k \geq 0$; $\{\zeta_i\}_{i=0}^\infty$ are independent non-negative random variables, and $\{\zeta_i\}_{i=1}^\infty$ are identically distributed.

Let the renewal process $(R_t, t \geq 0)$ with the renewal points $\{\theta_i\}_{i=0}^\infty$ be an embedded renewal process of the regenerative process $(X_t, t \geq 0)$, and let $(N_t, t \geq 0)$ be an embedded backward renewal process of the renewal process $(R_t, t \geq 0)$.

Once more we denote by $F(s) \stackrel{\text{def}}{=} \mathbf{P}\{\zeta_i \leq s\}$ for $i \geq 1$, and by $G(s) \stackrel{\text{def}}{=} \mathbf{P}\{\zeta_0 \leq s\}$; we assume that $G(s) = F_r(s)$ – see Denotation 4, and we suppose that *Key Condition* is satisfied.

A paired process $(V_t, t \geq 0) \stackrel{\text{def}}{=} ((X_t, N_t), t \geq 0)$ is a Markov regenerative process with the state space $(\overline{\mathcal{X}}, \sigma(\overline{\mathcal{X}})) \stackrel{\text{def}}{=} (\mathcal{X}, \sigma(\mathcal{X})) \times (\mathcal{R}, \sigma(\mathcal{R}))$, and the components X_t and N_t of the process $(V_t, t \geq 0)$ are dependent. Namely, there exists a conditional distribution

$$G_a(M) \stackrel{\text{def}}{=} \mathbf{P}\{X_t \in M | N_t = a\} = \mathbf{P}\{X_{\theta_k+a} \in M | \theta_k + a \leq \theta_{k+1}\},$$

where $M \in \sigma(\mathcal{X})$.

So, if we know the renewal times of the process $(R_t, t \geq 0)$ then it is possible to define the (conditional) distribution of the process $(X_t, t \geq 0)$ at any time (given the values of $\{\theta_i\}_{i=0}^\infty$), and we can propose the following method of construction of the successful coupling $((W_t, \widetilde{W}_t), t \geq 0)$ for the processes

$$(V_t, t \geq 0) = ((X_t, N_t), t \geq 0) \text{ and } (\widetilde{V}_t, t \geq 0) \stackrel{\text{def}}{=} ((\widetilde{X}_t, \widetilde{N}_t), t \geq 0).$$

First, it is possible to construct (on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ – see Denotation 5) the stationary successful coupling for the second parts of the processes $(V_t, t \geq 0)$ and $(\widetilde{V}_t, t \geq 0)$, i.e. for the processes $(N_t, t \geq 0)$ and

$(\tilde{N}_t, t \geq 0)$ by the schema described in Section 3. As a result, we obtain the process $(\mathcal{Z}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$.

Then we fix the time $\tilde{\tau}(r) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 : Z_t = \tilde{Z}_t \mid N_0 = r \right\}$; we do know that $\mathbf{E} \tilde{\tau}(r) < \infty$ (Lemma 4); and $\tilde{\tau}(r) = \theta_j = \vartheta_i$ for some i and j .

After that we can complete the process $(Z_t, t \geq 0)$ to the process $((Y_t, Z_t), t \geq 0)$ by construction (on some probability space $(\Omega', \mathcal{F}', \mathbf{P}')$) the random elements

$$\Theta_0 \stackrel{\text{def}}{=} \{Y_t, t \in [0, \theta_0] \mid \theta_0 = \zeta_0\} \stackrel{\mathcal{D}}{=} \{X_t, t \in [0, \theta_0] \mid \theta_0 = \zeta_0\}$$

and for $k = 1, 2, \dots, j$

$$\Theta_k \stackrel{\text{def}}{=} \{Y_t, t \in [\theta_{k-1}, \theta_k] \mid \theta_k - \theta_{k-1} = \zeta_k\} \stackrel{\mathcal{D}}{=} \{X_t, t \in [\theta_{k-1}, \theta_k] \mid \theta_k - \theta_{k-1} = \zeta_k\}.$$

Similarly, it is possible to complete the process $(\tilde{Z}_t, t \geq 0)$ to the process $((\tilde{Y}_t, \tilde{Z}_t), t \geq 0)$ by construction (on some probability space $(\Omega'', \mathcal{F}'', \mathbf{P}'')$) the random elements

$$\tilde{\Theta}_0 \stackrel{\text{def}}{=} \{\tilde{Y}_t, t \in [0, \tilde{\theta}_0] \mid \tilde{\theta}_0 = \tilde{\zeta}_0\} \stackrel{\mathcal{D}}{=} \{\tilde{X}_t, t \in [0, \tilde{\theta}_0] \mid \tilde{\theta}_0 = \tilde{\zeta}_0\}$$

and for $k = 1, 2, \dots, i$

$$\begin{aligned} \tilde{\Theta}_k &\stackrel{\text{def}}{=} \{\tilde{Y}_t, t \in [\tilde{\theta}_{k-1}, \tilde{\theta}_k] \mid \tilde{\theta}_k - \tilde{\theta}_{k-1} = \tilde{\zeta}_k\} \stackrel{\mathcal{D}}{=} \\ &\stackrel{\mathcal{D}}{=} \{\tilde{X}_t, t \in [\tilde{\theta}_{k-1}, \tilde{\theta}_k] \mid \tilde{\theta}_k - \tilde{\theta}_{k-1} = \tilde{\zeta}_k\}. \end{aligned}$$

At the time $\tilde{\tau}(r) = \theta_j = \vartheta_i$ the processes $(Z_t, t \geq 0)$ and $(\tilde{Z}_t, t \geq 0)$ have the same distribution (or even coincide in particular cases).

After the time $\tilde{\tau}(r) = \theta_j$ we can again construct (on some probability space $(\Omega''', \mathcal{F}''', \mathbf{P}''')$) the random elements

$$\Theta_k \stackrel{\text{def}}{=} \{Y_t, t \in [\theta_{k-1}, \theta_k] \mid \theta_k - \theta_{k-1} = \zeta_k\} \stackrel{\mathcal{D}}{=} \{X_t, t \in [\theta_{k-1}, \theta_k] \mid \theta_k - \theta_{k-1} = \zeta_k\}$$

for $k > j$, and we put $\tilde{\Theta}_{i+\ell} \stackrel{\text{def}}{=} \Theta_{j+\ell}$, $\ell > 0$.

Analogously, it is possible to construct the process

$$((W_t, \tilde{W}_t), t \geq 0) = ((Y_t, Z_t, \tilde{Y}_t, \tilde{Z}_t), t \geq 0) \stackrel{\mathcal{D}}{=} ((X_t, N_t, \tilde{X}_t, \tilde{N}_t), t \geq 0),$$

on the probability space

$$\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}\right) \stackrel{\text{def}}{=} (\Omega, \mathcal{F}, \mathbf{P}) \times (\Omega', \mathcal{F}', \mathbf{P}') \times (\Omega'', \mathcal{F}'', \mathbf{P}'') \times (\Omega''', \mathcal{F}''', \mathbf{P}'''),$$

and this process is a successful coupling for the processes $(V_t, t \geq 0) = ((X_t, N_t), t \geq 0)$ and $(\tilde{V}_t, t \geq 0) = ((\tilde{X}_t, \tilde{N}_t), t \geq 0)$ – here and hereafter we omit a detailed description of the construction of this process.

Now, from Corollary 1 and Corollary 2 we deduce:

Theorem 2. *Let $(X_t, t \geq 0)$ be a Markov regenerative process with the state space $(\mathcal{X}, \sigma(\mathcal{X}))$ which satisfies Key Condition, with the initial state $X_0 = x$, and suppose that $\mu_{0,K} \stackrel{\text{def}}{=} \mathbf{E}(\zeta_0)^K < \infty$, $\mu_K \stackrel{\text{def}}{=} \mathbf{E}(\zeta_1)^K < \infty$ for some $K \geq 1$.*

Then for all $t \geq 0$ and every $k \in [1, K]$ we derive an estimate

$$\|\mathcal{P}_t^x - \mathcal{P}\|_{TV} \leq 2\hat{C}(k, \zeta_0)t^{-k},$$

where $\mathcal{P}_t^r(M) \stackrel{\text{def}}{=} \mathbf{P}\{X_t \in M | X_0 = x\}$, $\mathcal{P}(M) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \mathcal{P}_t^x(M)$, $M \in \sigma(\mathcal{X})$, and ζ_0 is a first regeneration point of the process $(X_t, t \geq 0)$.

Theorem 3. *Let $(X_t, t \geq 0)$ be a Markov regenerative process with the state space $(\mathcal{X}, \sigma(\mathcal{X}))$ which satisfies Key Condition, with the initial state $X_0 = x$, and let $\mathbf{E}e^{a\zeta_0} = \varepsilon_{0,a} < \infty$, $\mathbf{E}e^{a\zeta_1} = \varepsilon_a < \infty$ for some $a > 0$.*

Then for all $t \geq 0$ and for all $\beta > 0$ such that $\tilde{\varepsilon}_\beta < 1$ we have

$$\|\mathcal{P}_t^x - \mathcal{P}\|_{TV} \leq 2\hat{\mathcal{C}}(\beta, \zeta_0)e^{-\beta t}.$$

Remark 16. Furthermore, it is possible to obtain a certain decrease in value of the constants $\hat{C}(k, \zeta_0)$ and $\hat{\mathcal{C}}(\beta, \zeta_0)$ using the properties of the cumulative distribution function $F(s)$ and determining more accurately the estimates in the calculations (8) and (9).

5.3 Application to the queueing theory.

The distribution of the period of the queueing regenerative process $(Q_t, t \geq 0)$ is often unknown in the queueing theory. However, the regeneration period

can be often split into two parts, in most cases we call them a busy period and an idle period. Furthermore, as a rule the idle period has a known non-discrete distribution. We suppose that the bounds for the moments of the busy period are also known. This queueing process has an *embedded alternating renewal process*, and it turns out not to be Markov, although the backward alternating renewal process defined similarly to the backward renewal process is Markov – see Definition 3. So, in this situation the queueing regenerative process has an *embedded backward alternating renewal process* $(A_t, t \geq 0)$.

The stationary coupling method can be applied to the backward alternating renewal process $(A_t, t \geq 0)$ in the case when one of its alternating renewal periods has the cumulative distribution function which satisfies *Key Condition*, and all alternating renewal periods have finite expectation. Moreover, it is possible to use the stationary coupling method for to the backward alternating renewal process in some cases when the alternating periods of one regenerative period are dependent. The description of the stationary coupling method for the backward alternating process will be presented in the next publications.

So, we can find the bounds for the convergence of the backward renewal process $(A_t, t \geq 0)$ by using the stationary coupling method. Then, using the argumentation of Section 5.2, we can verify that the bounds for the process $(A_t, t \geq 0)$ appear to be true for a complete process $((Q_t, A_t), t \geq 0)$ and for the process $(Q_t, t \geq 0)$.

Besides, if the bounds of moments of a busy period are also known, we can apply our construction to embedded alternating renewal process after some modification.

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